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STRESSES IN SOME ANISOTROPIC MATERIALS DUE TO INDENTATION AND SLIDING

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Abstract—This paper is concerned with the stress field in Hertzian contact of parallel cylinders composed of some general anisotropic materials, and of the Hertzian contact of transversely isotropic spherical bodies. Explicit analytical expressions are found for the stress components in each case. Numerical results indicate that the maximum shear stress distribution may deviate significantly from that of the isotropic case.

1. INTRODUCTION

THE analysis of deformations of two elastic bodies in contact with each other usually rests upon the Hertzian contact model in linear elasticity. The application of this model to engineering problems has met with success (e.g. MacGregor [1]). The Hertz contact problem considers two bodies characterized by two principal radii of curvature in the contact region. We shall restrict ourselves to contact of parallel cylinders (plane strain), discs (plane stress) and spheres. Usually the size of the area of contact is small in comparison to the radius of curvature of the bodies in contact, therefore one may assume that one of the bodies can be replaced by an elastic semi-infinite space.

When the material is isotropic, its elastic property is governed by two elastic constants. In the case of an anisotropic body, there are more of these constants. Plane contact problems in plane anisotropic elasticity have been discussed in the books by Galin [2], and by Green and Zerna [3]. In both these books, the close similarity between isotropic and anisotropic problems has been emphasized. However, due to the conciseness of the treatment in Green and Zerna and an apparent oversight in Galin, it has not been brought out that in certain important practical situations the stress functions for the isotropic and anisotropic materials are of the same form. Moreover, in these cases the form of the expressions for pressure distribution underneath the punch is independent of the material properties of the half space. We shall discuss these points and illustrate our results with the Hertz contact problem. It will be shown that in these cases the normal displacements at the contact surface are of the form $(2a^2-x^2)$ and the pressure distributions are of the form $(a^2-x^2)^{1/2}$ regardless of whether the material is isotropic or anisotropic. In the latter case the stress distribution inside the elastic body is generally not symmetric, although the external load is symmetric; the only exception is when the material is orthotropic and one of its axes of symmetry coincides with the axis of loading.

An analogous mixed boundary value problem is to specify a tangential displacement over a region at the surface and no stress elsewhere on the surface. The stress functions for the isotropic and anisotropic materials are again found to be the same, and the shear stress distribution where the displacement is specified is of the same form. Finally, we write out the solution to the Hertzian sliding contact problem with the frictional effect included.

In practice, many of the crystals of interest possess only three (cubic crystals) or five (hexagonal crystals) elastic constants. However, usually the axis of loading does not coincide with one of the axes of the crystals and one needs to use the general formulation contained in this work.

We have computed the plane stress fields of some anisotropic materials under indentation with and without sliding. A detailed description of analysis and numerical results of the corresponding isotropic problem may be found in a paper by Poritsky [4]. The elastic constants used here are those of copper and zinc. We tried different orientations of the crystal axes with respect to the geometric axis of loading. We are interested mainly in the maximum shear stress. For an isotropic material under indentation only, it is known that the greatest value of the maximum shear stress occurs on the axis of symmetry at a depth of about four tenths of the contact width. It has been generally assumed that for an orthotropic material in which the material and geometric axis coincide, the stress would still be situated on the axis of symmetry at some distance below the contact surface. The above assumption is found to be not true. The maximum value may lie outside the axis of symmetry, or it may lie on the contact surface. In other words, the stress distribution for an anisotropic material may depart quite radically from that for an isotropic material.

In three-dimensional elasticity, the transversely isotropic material appears to be the only anisotropic material for which analytical approach is possible. In view of the numerical results found in plane problems, it is important to investigate the cases of the contact of spherical bodies composed of this type of anisotropic materials. From a knowledge of the isotropic solution (Huber [5], and Hamilton and Goodman [6]), closed-form solutions for indentation and sliding are obtained. Using the indentation result, we have computed the numerical values of the stress components in several anisotropic materials. It is found that in some anisotropic cases the maximum shear stress distribution can be quite different from that in the corresponding isotropic case. In particular, the point of maximum shear stress may not be situated on the axis of symmetry. This result shows that a knowledge of the stress field along the axis of symmetry only is not adequate for the determination of the greatest maximum shear stress.

It is not, of course, implied that maximum shear stress is the yield or fracture criterion in anisotropic materials. One would expect that in different types of anisotropic materials such as crystals, fiber-reinforced materials, or soil, the mechanism for failures would be quite different. In crystals, for instance, the critical values to look for are probably the shear stresses on the slip planes. The important lesson to be learned from this study is that stress distributions in isotropic and anisotropic materials may turn out to be quite different; and intuitions and conclusions inferred from the former may not be applicable to the latter. Whenever possible, one should compute the stress field for each case individually. For this problem we have provided the analytical results for the purpose.

2. PLANE ANISOTROPIC PROBLEMS

A. *Basic equations*

Hooke's law for an anisotropic body in plane elasticity has the following form (Green and Zerna [3]):

$$e_{xx} = \frac{\partial u_x}{\partial x} = s_{11}\sigma_{xx} + s_{12}\sigma_{yy} + s_{16}\sigma_{xy},$$
$$e_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} = s_{16}\sigma_{xx} + s_{26}\sigma_{yy} + s_{66}\sigma_{xy}, \tag{1}$$
$$e_{yy} = \frac{\partial u_y}{\partial y} = s_{12}\sigma_{xx} + s_{22}\sigma_{yy} + s_{26}\sigma_{xy}.$$

For plane stress, the s_{ij}'s are equal to their respective elastic compliances, the β_{ij}'s. In the case of the plane strain, the s_{ij}'s are combinations of elastic compliances, as can be easily shown.

In the case when the material is isotropic, the s_{ij}'s may be expressed in terms of the Young's modulus (E) and Poisson's ratio (ν). For plane stress they are

$$\begin{aligned} s_{11} &= s_{22} = 1/E, \qquad s_{26} = s_{16} = 0, \\ s_{12} &= -\nu/E, \qquad s_{66} = 2(1+\nu)/E. \end{aligned} \tag{2}$$

For plane strain they are

$$\begin{aligned} s_{11} &= s_{22} = (1-\nu^2)/E, \qquad s_{26} = s_{16} = 0, \\ s_{12} &= -(\nu+\nu^2)/E, \qquad s_{66} = 2(1+\nu)/E. \end{aligned} \tag{3}$$

The components of stress should satisfy the equilibrium conditions.

$$\begin{aligned} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} &= 0 \\ \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} &= 0. \end{aligned} \tag{4}$$

Combining equations (1) and (2), it can be shown that the stresses and displacements in an anisotropic body can be expressed in terms of two functions of complex variables

$$z_1 = x + \mu_1 y, \qquad z_2 = x + \mu_2 y. \tag{5}$$

μ_1 and μ_2 and the complex conjugates $\bar{\mu}_1$ and $\bar{\mu}_2$ are the roots of the equation.

$$s_{11}\mu^4 - 2s_{16}\mu^3 + (2s_{12} + s_{66})\mu^2 - 2s_{26}\mu + s_{22} = 0. \tag{6}$$

The stresses and displacements are expressed in terms of two complex stress functions $\phi_1(z_1)$ and $\phi_2(z_2)$,

$$\sigma_{xx} = \mu_1^2\phi_1'(z_1) + \bar{\mu}_1^2\bar{\phi}_1'(\bar{z}_1) + \mu_2^2\phi_2'(z_2) + \bar{\mu}_2^2\bar{\phi}_2'(\bar{z}_2). \tag{7}$$
$$\sigma_{xy} = -\mu_1\phi_1'(z_1) - \bar{\mu}_1\bar{\phi}_1'(\bar{z}_1) - \mu_2\phi_2'(z_2) - \bar{\mu}_2\bar{\phi}_2'(\bar{z}_2). \tag{8}$$
$$\sigma_{yy} = \phi_1'(z_1) + \bar{\phi}_1'(\bar{z}_1) + \phi_2'(z_2) + \bar{\phi}_2(\bar{z}_2). \tag{9}$$
$$u_x = p_1\phi_1(z_1) + \bar{p}_1\bar{\phi}_1(\bar{z}_1) + p_2\phi_2(z_2) + \bar{p}_2\bar{\phi}_2(\bar{z}_2). \tag{10}$$
$$u_y = q_1\phi_1(z_1) + \bar{q}_1\bar{\phi}_1(\bar{z}_1) + q_2\phi_2(z_2) + \bar{q}_2\bar{\phi}_2(\bar{z}_2). \tag{11}$$

In equations (10) and (11), the constants p_1, p_2 and q_1, q_2 are given by the following expressions

$$\begin{aligned} p_1 &= s_{11}\mu_1^2+s_{12}-s_{16}\mu_1, \\ p_2 &= s_{11}\mu_2^2+s_{12}-s_{16}\mu_2, \\ q_1 &= (s_{12}\mu_1^2+s_{22}-s_{26}\mu_1)/\mu_1, \\ q_2 &= (s_{12}\mu_2^2+s_{22}-s_{26}\mu_2)/\mu_2. \end{aligned} \tag{12}$$

When the material is isotropic, the complex variable formulation for stresses and displacements are well known. We have

$$\begin{aligned} \sigma_{xx}+\sigma_{yy} &= 2[\phi'(z)+\bar{\phi}'(\bar{z})], \\ \sigma_{yy}-\sigma_{xx}+2i\sigma_{xy} &= 2[\bar{z}\phi''(z)+\psi'(z)], \\ 2G(u_x+iu_y) &= K\phi(z)-z\bar{\phi}'(\bar{z})-\bar{\psi}(\bar{z}), \\ K &= 3-4\nu \quad \text{(plane strain)}, \\ K &= (3-\nu)/(1+\nu) \quad \text{(plane stress)}. \end{aligned} \tag{13}$$

where G is the shear modulus.

B. *Load normal to surface of the half-plane*

We shall consider the medium occupying the half-plane $y \geq 0$, which we shall call S^+. A schematic diagram of the loading is shown in Fig. 1. The boundary to the half-plane is

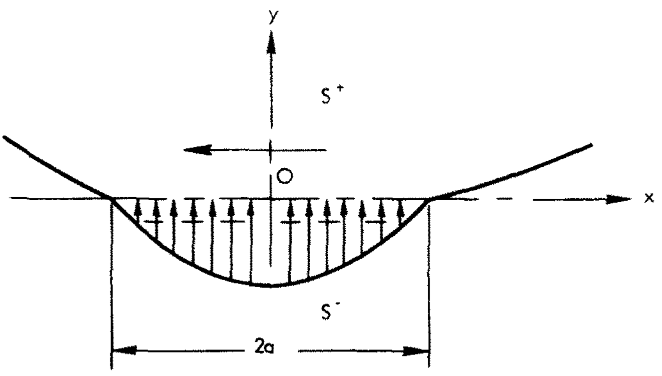

FIG. 1. Schematic view of contact and sliding. The elastic body is the upper half-space. The normal force and tangential force are both in the positive y and x directions.

O_x. We shall assume that there is no frictional force between the punch and the half-plane, so that the resultant force is normal to the surface of the half-plane. The boundary conditions on the surface are

$$\begin{aligned} \sigma_{xy} &= 0 \qquad \text{everywhere on } O_x \\ \sigma_{yy} &= 0 \qquad \text{on } O_x \text{ outside segment } C \\ u_y &= F(x) \qquad \text{on segment } C \end{aligned} \tag{14}$$

where segment C is the contact area beneath the punch and half-space. $F(x)$ is a known function defining the shape of the punch. Segment C will be taken as the segment

$$-a < x < a.$$

1. *Anisotropic medium.* We shall assume that the functions $\phi_1(z)$ and $\phi_2(z)$ in equations (7) to (11) are related by

$$\begin{aligned} \phi_1(z) &= \mu_2\phi(z)/(\mu_2-\mu_1), \\ \phi_2(z) &= \mu_1\phi(z)/(\mu_1-\mu_2). \end{aligned} \tag{15}$$

Thus the condition that shear stress is zero on the surface $y = 0$ is now satisfied, and on this surface σ_{yy}, $\partial u_y/\partial x$ are

$$\sigma_{yy} = [\phi'(z)+\bar{\phi}'(\bar{z})], \tag{16}$$

$$\frac{\partial u_y}{\partial x} = -\frac{q_1\mu_2-q_2\mu_1}{\mu_1-\mu_2}\phi'(z)-\frac{\bar{q}_1\bar{\mu}_2-\bar{q}_2\bar{\mu}_1}{\bar{\mu}_1-\bar{\mu}_2}\bar{\phi}'(\bar{z}). \tag{17}$$

By using equations (12) and (6) the term $(q_1\mu_2-q_2\mu_1)/(\mu_1-\mu_2)$ is found to be always imaginary. Hence equation (17) can be written as

$$\frac{\partial u_y}{\partial x} = -iS_1\{\phi'(z)-\bar{\phi}(\bar{z})\}, \tag{18}$$

where S_1 is defined as

$$S_1 = \mathrm{Im}\left[\frac{q_1\mu_2-q_2\mu_1}{\mu_1-\mu_2}\right] = -\frac{s_{22}}{2i}\left[\frac{1}{\mu_1}-\frac{1}{\bar{\mu}_1}+\frac{1}{\mu_2}-\frac{1}{\bar{\mu}_2}\right]. \tag{19}$$

μ_1 and μ_2 are found from equation (6).

The boundary conditions described by equations (14) now give the equations governing the function $\phi'(z)$,

$$\begin{aligned} \mathrm{Re}\{\phi'(z)\} &= 0, \qquad |x| > a, \qquad y = 0, \\ \mathrm{Im}\{\phi'(z)\} &= \frac{F'(x)}{2S_1}, \qquad |x| < a, \qquad y = 0. \end{aligned} \tag{20}$$

This is a particular case of the Hilbert problem and the general solution to it is known to be (Mikhlin [7], p. 311),

$$\phi'(z) = \frac{1}{2\pi S_1\sqrt{(a^2-z^2)}}\int_{-a}^{a}\frac{F'(t)\sqrt{(a^2-t^2)}\,dt}{t-z}+\frac{M}{\sqrt{(a^2-z^2)}}. \tag{21}$$

M is a real constant and is zero where no sharp corners are in contact between the punch and the elastic half space. For our purpose we shall restrict ourselves to smooth punches and take M identically zero. The pressure distribution exerted by the punch is

$$\sigma_{yy} = 2\mathrm{Re}\{\phi'(x)\}, \tag{22}$$

and the resultant normal force is

$$Y = -\frac{1}{S_1}\int_{-a}^{a}\frac{tF'(t)\,dt}{\sqrt{(a^2-t^2)}}. \tag{23}$$

Combining equations (21) and (23) to eliminate S_1, we obtain,

$$\phi'(z) = -\frac{Y}{2\pi\sqrt{(a^2-z^2)}}\int_{-a}^{a}\frac{F'(t)\sqrt{(a^2-t^2)}}{t-z}\mathrm{d}t \Big/ \int_{-a}^{a}\frac{tF'(t)\,\mathrm{d}t}{\sqrt{(a^2-t^2)}}. \tag{24}$$

An examination of equations (22) and (15) shows that the pressure underneath a smooth punch as well as the form of the stress functions are independent of the elastic property of the half space.

It is reasonable to expect that the above results are also true for isotropic elasticity where the governing equation is biharmonic. For the sake of completeness we shall give a brief sketch of the same physical problem.

2. *Isotropic medium.* If we set in equations (13), the following relation

$$z\phi'(z)+\psi(z) = \phi(z) \tag{25}$$

then the requirement of vanishing of shear stress is satisfied. Along the boundary O_x

$$\begin{aligned} \sigma_{yy} &= 2\,\mathrm{Re}\{\phi'(z)\}, \\ 2G\frac{\partial u_y}{\partial x} &= (1+k)\,\mathrm{Im}\{\phi'(z)\}. \end{aligned} \tag{26}$$

The boundary conditions described by equation (14) now lead to

$$\begin{aligned} \mathrm{Re}\{\phi'(z)\} &= 0, & |x| > a, \quad & y = 0; \\ \mathrm{Im}\{\phi'(z)\} &= \frac{2G}{1+k}F'(z), & |x| < a, \quad & y = 0. \end{aligned} \tag{27}$$

This is again a particular case of the Hilbert problem as in equation (20), and the solution is of the same form as is given by equation (21),

$$\phi'(z) = \frac{2G}{\pi(1+k)\sqrt{(a^2-z^2)}}\int_{-a}^{a}\frac{F'(t)\sqrt{(a^2-t^2)}\,\mathrm{d}t}{t-z}+\frac{M}{\sqrt{(a^2-z^2)}} \tag{28}$$

Again we shall assume that the shape of the punch is smooth and assume H equal to zero. The resultant force is

$$Y = -\frac{4G}{1+k}\int_{-a}^{a}\frac{tF'(t)\,\mathrm{d}t}{\sqrt{(a^2-t^2)}}. \tag{29}$$

Equations (28) and (29) should be compared with equations (19) and (21). It appears that they would be the same pair if the constant S_1 were replaced by $(1+k)/(4G)$.

3. *Example: Hertzian indentation problem.* Assume that the elastic plane is indented by a circular shaped punch of radius R expressed as

$$F(x) = d-x^2/2R \tag{30}$$

Then

$$F'(x) = -x/R$$

and the total indentation force is

$$Y = \pi a^2/(2S_1R), \qquad \text{(anisotropic material)}; \tag{31}$$

$$Y = 2\pi\mu a^2/[(1+k)R], \quad \text{(isotropic material)}. \tag{32}$$

The stress function is

$$\phi'(z) = \frac{iY}{\pi a^2}\{\sqrt{(z^2-a^2)}-z\}, \tag{33}$$

and the traction along the boundary $y = 0$ is

$$\begin{aligned} \sigma_{yy} &= -\frac{2Y}{\pi a}\left(1-\frac{x^2}{a^2}\right)^{1/2} \qquad |x| < a, \qquad y = 0; \\ \sigma_{xy} &= \sigma_{yy} = 0, \qquad |x| > a, \qquad y = 0. \end{aligned} \tag{34}$$

Equations (33) and (34) are valid for both the isotropic and anisotropic materials. This result illustrates that if $S_1 = (1+k)/(4G)$, then the penetration of the punch and the pressure distribution between the punch and the half-space are identical. The maximum contact pressure which we shall call q_0 is

$$q_0 = 2Y/\pi a. \tag{35}$$

Usually in contact problems we do not know a, the half-width of the contact area, but we know R which is the radius of curvature of the punch. Combining equations (30) and (33), eliminating a, we find the formula for maximum contact pressure in terms of the load and radius of curvature,

$$q_0 = \sqrt{\left(\frac{Y}{2S_1\pi R}\right)}. \tag{36}$$

In the case of two bodies in contact, the generalization is completely analogous to the isotropic situation, and the half contact width is

$$a = 2\sqrt{\left[\frac{YR_1R_2(S_1+S'_1)}{R_1+R_2}\right]}. \tag{37}$$

where S_1 and S'_1 are from equation (19) for the two elastic materials in contact. The constant S_1 is a measure of the compliance of the material under indentation. For an anisotropic material, the value of S_1 also depends upon the orientation of the crystal axis with respect to surface of the half-plane.

C. *Load tangential to the surface of the half-plane*

We shall again consider the half-space $y > 0$, and prescribe the boundary condition along the line $y = 0$, i.e. the O_x axis. We now assume that, at the surface of the half-space shear stress is prescribed over part of the boundary, and tangential displacement over the other part of the boundary, the normal stress being zero everywhere. Following the notations of equation (4), we have along $y = 0$,

$$\begin{aligned} \sigma_{yy} &= 0; \\ \sigma_{xy} &= 0, \qquad |x| > a; \\ u_x &= H(x), \qquad |x| < a. \end{aligned} \tag{38}$$

where $H(x)$ is a known function defining the tangential surface displacement.

We shall assume in equations (7)–(11) that

$$\begin{aligned} \phi_1(z_1) &= \psi(z_1)/(\mu_2-\mu_1), \\ \phi_2(z_2) &= \psi(z_2)/(\mu_1-\mu_2). \end{aligned} \tag{39}$$

The condition that normal stress vanishes on the plane surface is now satisfied. And on this surface $y = 0$

$$\sigma_{xy} = \psi'(z)+\bar{\psi}'(\bar{z}), \tag{40}$$

$$\frac{\partial u_x}{\partial x} = \frac{p_1-p_2}{\mu_1-\mu_2}\psi'(z)+\frac{\bar{p}_1-\bar{p}_2}{\bar{\mu}_1-\bar{\mu}_2}\bar{\psi}'(\bar{z}). \tag{41}$$

By using equations (12) and (6), it is found that the term $(p_2-p_1)/(\mu_2-\mu_1)$ is always imaginary. Equation (41) can now be written as, on $y = 0$

$$\frac{\partial u_x}{\partial x} = -iS_2\{\psi'(z)-\bar{\psi}'(\bar{z})\}, \tag{42}$$

where S_2 is real and defined as

$$S_2 = -\frac{s_{11}}{2i}(\mu_1+\mu_2-\bar{\mu}_1-\bar{\mu}_2). \tag{43}$$

The boundary conditions defined by equations (38) now give the equations governing $\psi'(z)$:

$$\begin{aligned} \mathrm{Re}\{\psi'(z)\} &= 0, \qquad |x| > a, \qquad y = 0; \\ \mathrm{Im}\{\psi'(z)\} &= \frac{H'(x)}{2S_2}, \qquad |x| < a, \qquad y = 0. \end{aligned} \tag{44}$$

The above set of equations is exactly the same as in equation (20), except that $H'(x)/2S_2$ is replaced by $F'(x)/2S_1$, and the general solution is

$$\psi'(z) = \frac{1}{2\pi S_2\sqrt{(a^2-z^2)}}\int_{-a}^{a}\frac{H'(t)\sqrt{(a^2-t^2)}\,\mathrm{d}t}{t-z}+\frac{N}{\sqrt{(a^2-z^2)}}. \tag{45}$$

In the above equation N is a real constant which may be set to zero when shear stress is finite on the surface of the half-plane. The resultant transverse force X is

$$X = -\frac{1}{S_2}\int_{-a}^{a}\frac{tH'(t)\,\mathrm{d}t}{\sqrt{(a^2-t^2)}} \tag{46}$$

and the shear stress distribution between $-a < x < a$, on $y = 0$ is

$$\sigma_{xy} = 2\,\mathrm{Re}\{\psi'(z)\}. \tag{47}$$

We shall not go through the analysis for the isotropic material here. We can prove that for the isotropic material the relationships between shear stress, tangential displacement and transverse force are described by equations (45), (46) and (47) if S_2 is replaced by $(1+k)/(4G)$. In other words, on the surface $y = 0$, if we prescribe the condition that normal stress is zero everywhere, and shear stress is zero except over the section where tangential displacement is prescribed, then the shear stress distribution over this section is the same as in the isotropic case except for a multiplicative constant S_2. This constant S_2 can of course be eliminated from the equations if the resultant force x is known.

D. *Indentation with transverse motion* (*sliding contact*)

When a punch is pressed down onto the half-space and then moved in a transverse direction, the usual assumption of coulomb friction is to have

$$\sigma_{xy} = \pm\rho\sigma_{yy}, \tag{48}$$

where ρ is the coefficient of friction. The plus-or-minus sign indicates that the direction of motion is positive or negative along the x-axis. The exact approach to this problem is to replace the first boundary condition from equations (14) by equation (48), and proceed to analyze that problem.

The generally accepted approximate method ([13], [6], [7]) to study indentation with sliding consists of two steps. The first step is to solve the punch problem without friction, as we have done in Section B. From a knowledge of the pressure at the surface, the shear stress that would occur upon sliding is found through equation (48). The second step is to solve the elasticity problem with the prescribed shear stress. The superposition of the two solutions gives the approximate solution to the problem of indentation with sliding.

The procedure described above can be very simply formulated using the results in Sections B and C. Comparing equations (22) and (47), it is seen that the condition (48) is satisfied if the stress functions $\phi(z)$ and $\psi(z)$ are related by

$$\psi'(z) = \pm\rho\phi'(z). \tag{49}$$

In other words, for any indentation problem described by equations (14), if the stress functions for contact without sliding are given by equation (15), then the stress functions for contact with sliding are given by

$$\begin{aligned} \phi_1'(z_1) &= (\mu_2 \pm \rho)\phi'(z_1)/(\mu_2-\mu_1) \\ \phi_2'(z_2) &= (\mu_1 \pm \rho)\phi'(z_2)/(\mu_1-\mu_2). \end{aligned} \tag{50}$$

In the case of Hertzian contact, the stress function $\phi'(z)$ has been given in equation (33). After the stress functions ϕ_1, and ϕ_2 are found, the stress field throughout the half-space is obtained using equations (7), (8) and (9). The stresses are

$$\begin{aligned} \sigma_{xx} &= \frac{2Y}{\pi a^2}\mathrm{Re}\left\{\frac{\mu_1^2(\mu_2-\rho)[(z_1^2-a^2)^{1/2}-z_1]-\mu_2^2(\mu_1-\rho)[(z_2^2-a^2)^{1/2}-z_2]}{\mu_2-\mu_1}\right\} \\ \sigma_{yy} &= \frac{2Y}{\pi a^2}\mathrm{Re}\left\{\frac{(\mu_2-\rho)[(z_1^2-a^2)^{1/2}-z_1]-(\mu_1-\rho)[(z_2^2-a^2)^{1/2}-z_2]}{\mu_2-\mu_1}\right\} \\ \sigma_{xy} &= \frac{2Y}{\pi a^2}\mathrm{Re}\left\{\frac{\mu_1(\mu_2-\rho)[(z_1^2-a^2)^{1/2}-z_1]-\mu_2(\mu_1-\rho)[(z_2^2-a^2)^{1/2}-z_2]}{\mu_2-\mu_1}\right\}. \end{aligned} \tag{51}$$

In equations (51) we have assumed that the transverse motion is along the positive x axis. The value of ρ will have to be negative if the motion is in the reverse direction. The values of μ_1 and μ_2 are obtained from equation (6).

E. *Numerical results*

We have computed the stress fields in several anisotropic materials, induced by Hertzian contact without sliding, i.e. using equation (51) with $\rho = 0$. We assume that the anisotropic materials are made up of copper or zinc crystals oriented at different angles with respect to the geometric axis of loading. The values of the elastic constants of the two crystals have been taken from a survey paper by Huntington [8]. The numerical data will be presented in the form of contour lines of constant maximum shear stress in the plane of loading. The dimensions and load are normalized such that the width of the contact area is 2, and the maximum pressure within the contact area is unity.

Copper is a cubic crystal, whose elastic properties are described by three elastic constants. In Figs. 2–6 we show the contours of maximum shear stress within the copper crystals at several different orientations. In Fig. 2 one of the crystal axes coincides with the O_y axis, the direction of loading. We note that the contour lines are symmetric, and the maximum shear stress occurs at two symmetric points away from the axis. Along the axis, the local maximum is about 0.247, while the actual maximum over the whole half-space is 0.314. Figures 3, 4, and 5 show the contours when the crystal has been rotated by 10°,

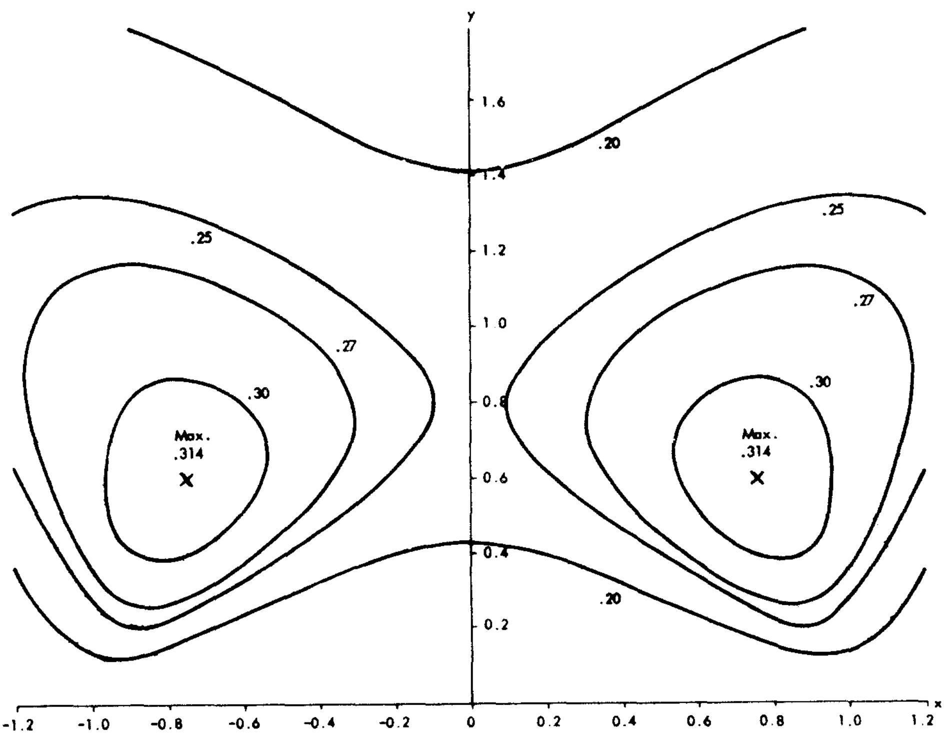

FIG. 2

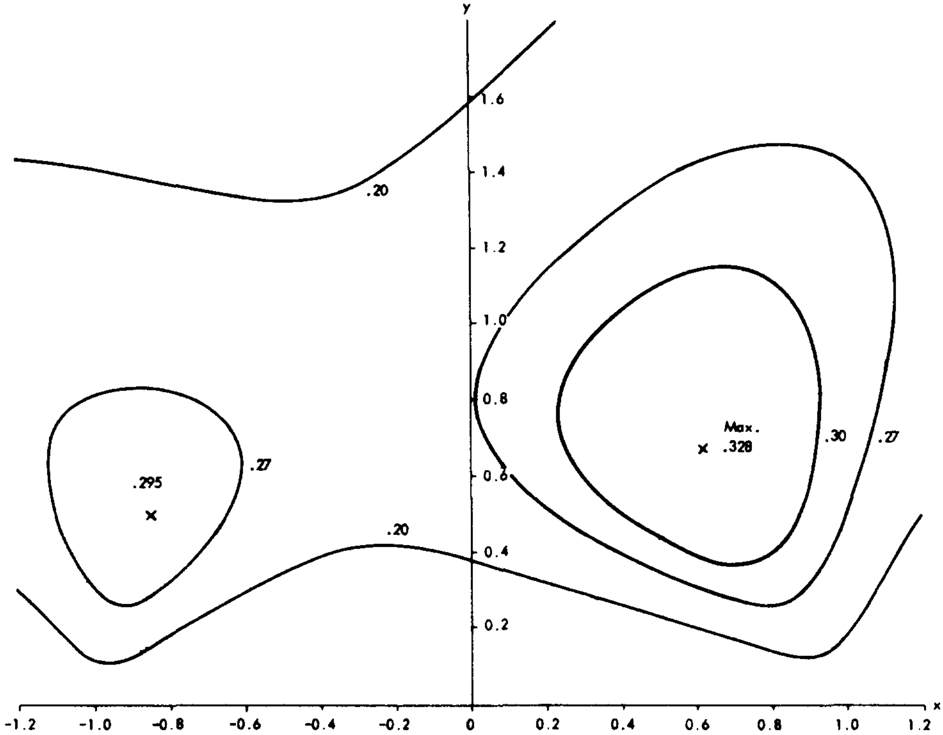
y
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
.20
.27
.295
Max.
.328
.30
.27
.20
-1.2
-1.0
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1.0
1.2
x

FIG. 3

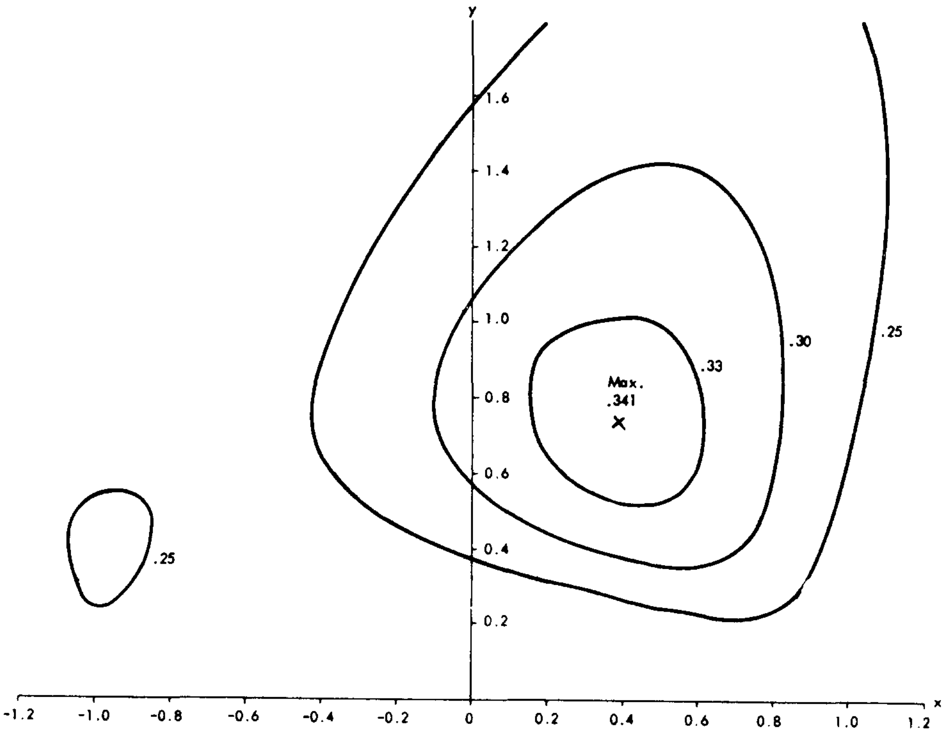
y
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
.25
.30
.33
Max.
.341
.25
-1.2
-1.0
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1.0
1.2
x

FIG. 4

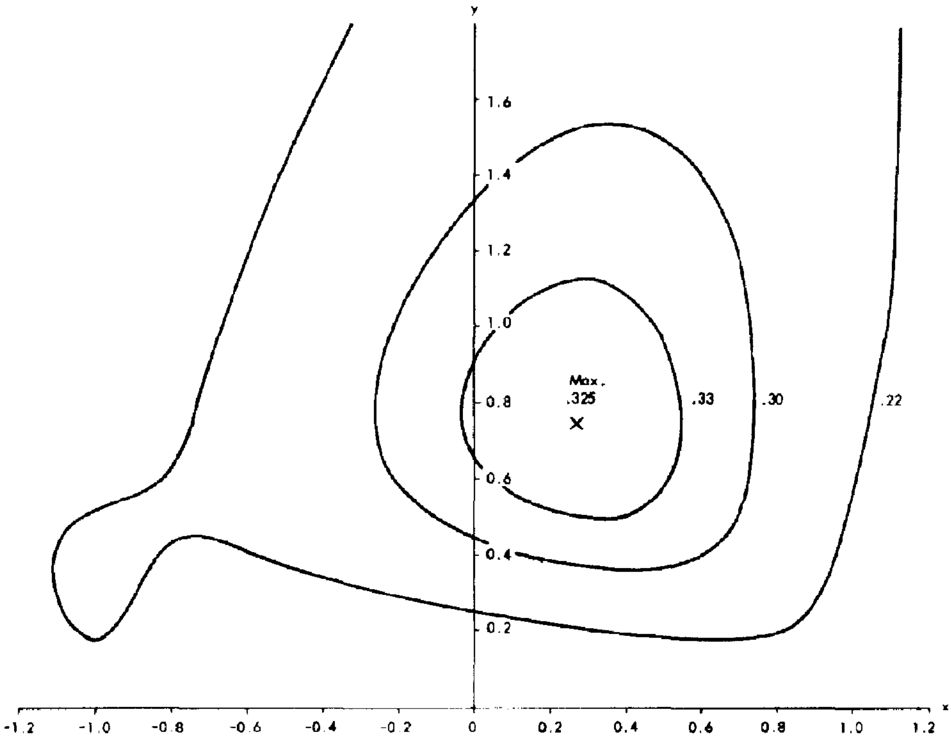
y
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
Max.
.325
.33
.30
.22
x
-1.2
-1.0
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1.0
1.2

FIG. 5

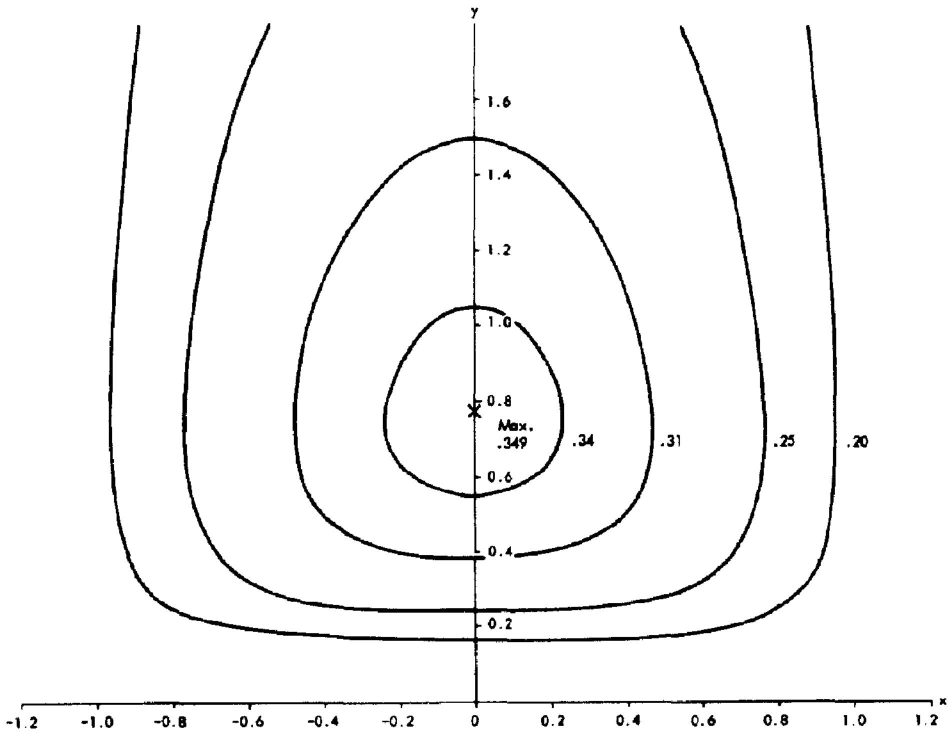
y
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
Max.
.349
.34
.31
.25
.20
x
-1.2
-1.0
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1.0
1.2

FIG. 6

22.5°, and 30°, respectively. Note that the position of the maximum value gradually moves to the center as the crystal is rotated. At 45° the contour lines are again symmetric with the O_y axis as may be expected for cubic crystals. Figure 6 shows that for this case the maximum is situated on the axis.

Next we consider the zinc crystal oriented at some angle with respect to the O_y axis. Zinc is an hexagonal crystal whose elastic property can be described by five elastic constants, and possesses only one axis of symmetry. We first line up the material axis with the O_y axis, and Fig. 7 shows the maximum shear stress contour lines. We then rotate the crystal by 45°, 60°, and 90°. Figures 8, 9, and 10 show the maximum shear stress plots for these three cases respectively. Note that in the last case, there are two local maxima at two symmetric points, while the actual maximum is at the contact surface.

We have also studied a number of cases with the effect of sliding included, i.e. $\rho \neq 0$ in equation (51). It does not seem profitable to present a profusion of data on the effect of sliding for different anisotropic materials. It appears that in anisotropic indentation problems, the greatest value of maximum shear stress is still located in the vicinity of the contact area. However, if one wishes a detailed description of the stress field, one must compute each case individually, since the results for one material may differ considerably from the results for another material.

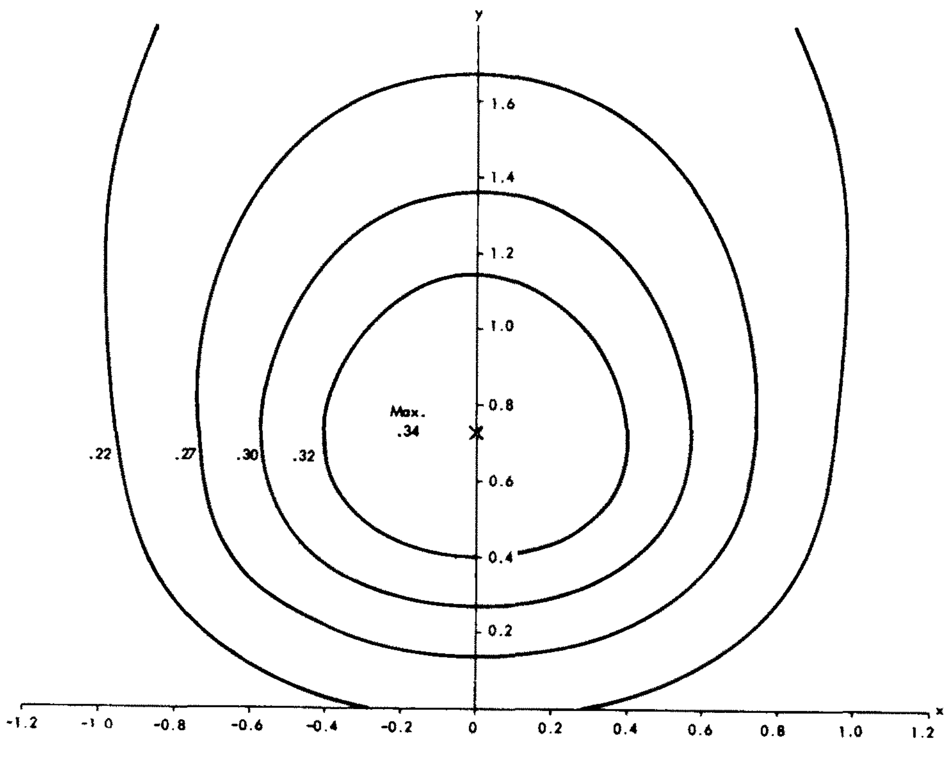

FIG. 7

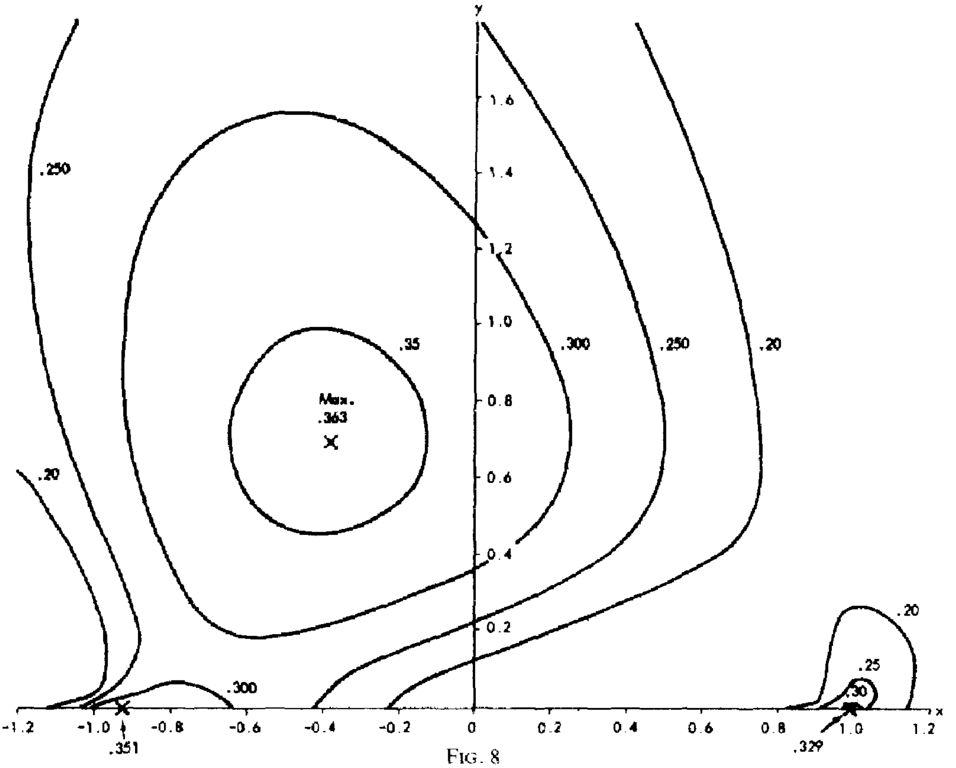
y
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
.250
.20
.35
Max.
.363
.300
.250
.20
.20
.25
.30
.300
.351
.329
-1.2
-1.0
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1.0
1.2
x

FIG. 8

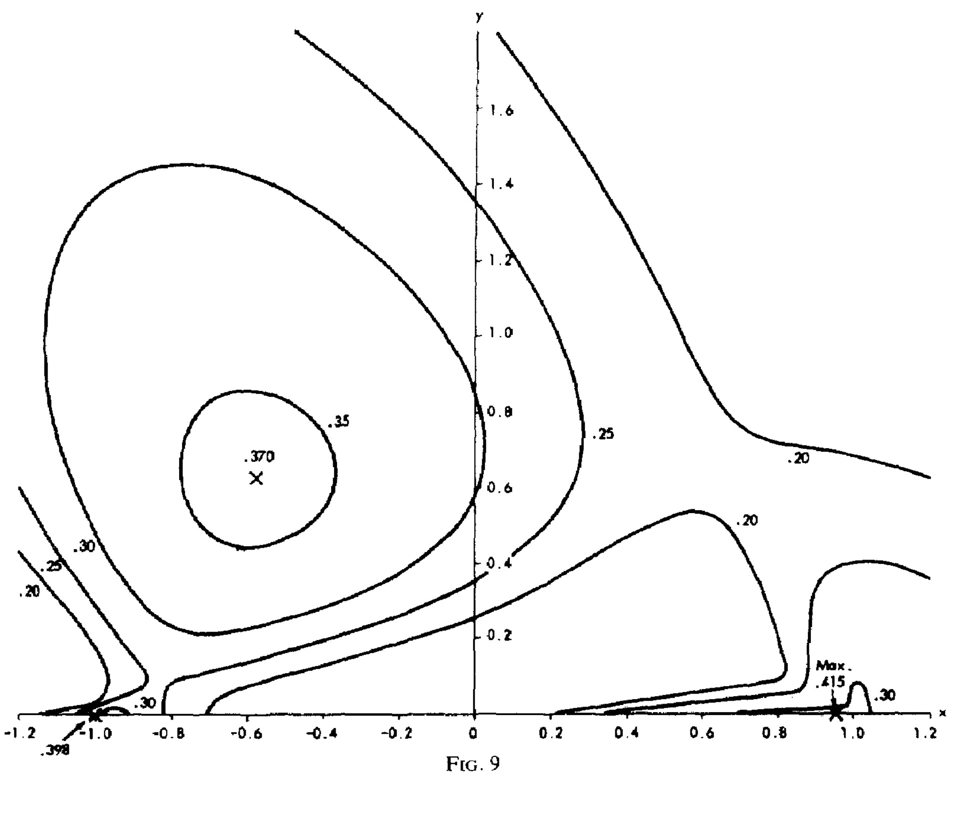
y
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
.35
.370
.25
.20
.20
.30
.25
.20
.30
.398
Max.
.415
.30
-1.2
-1.0
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1.0
1.2
x

FIG. 9

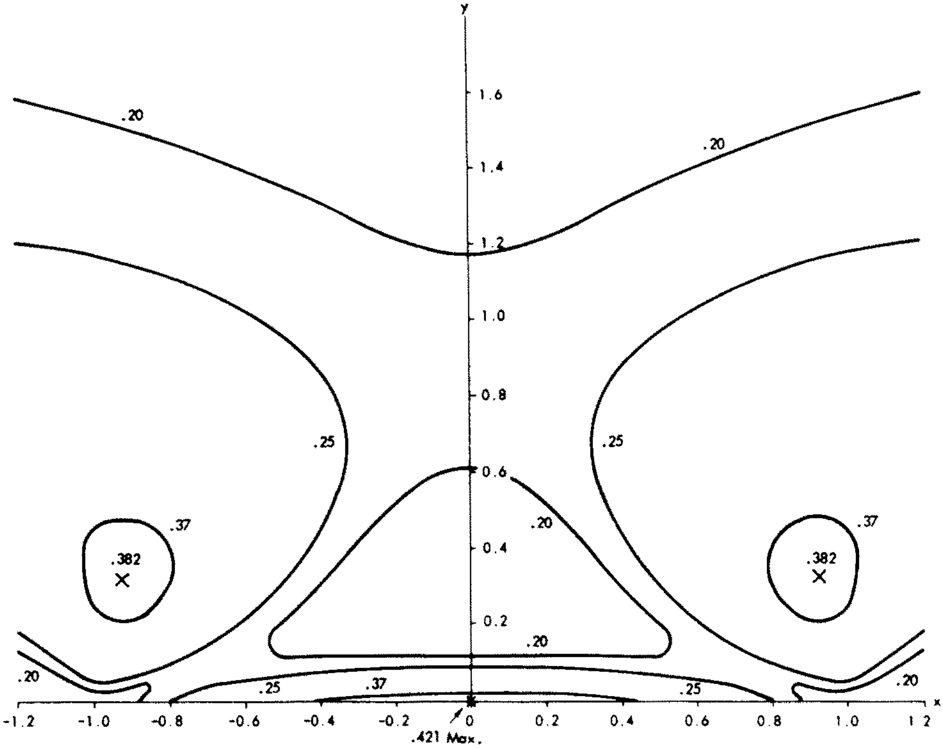

FIG. 10

3. THREE-DIMENSIONAL PROBLEMS IN TRANSVERSELY ISOTROPIC MATERIALS

A. *Basic equations*

In a transversely isotropic elastic solid the stress–strain relationships are given by

$$
\begin{aligned}
\sigma_{rr} &= c_{11}e_{rr}+c_{12}e_{\theta\theta}+c_{13}e_{zz},\\
\sigma_{\theta\theta} &= c_{12}e_{rr}+c_{11}e_{\theta\theta}+c_{13}e_{zz},\\
\sigma_{zz} &= c_{13}e_{rr}+c_{13}e_{\theta\theta}+c_{33}e_{zz},\\
\sigma_{\theta z} &= c_{44}e_{\theta z},\\
\sigma_{rz} &= c_{44}e_{rz},\\
\sigma_{r\theta} &= \tfrac{1}{2}(c_{11}-c_{12})e_{r\theta}.
\end{aligned}
\tag{52}
$$

In the above equations, (r, θ, z) is a set of cylindrical coordinates with the z axis parallel to the material axis. The five c_{ij}'s are the elastic constants. We shall now define some dimensionless parameters to be used in the potential function solutions. Let ν_1 and ν_2 be two roots of the algebraic equation

$$c_{11}c_{44}\nu^2+[c_{13}(2c_{44}+c_{13})-c_{11}c_{33}]\nu+c_{33}c_{44} = 0, \tag{53}$$

and

$$\nu_3 = 2c_{44}/(c_{11}-c_{12}). \tag{54}$$

It may be proved from the requirement of the positive definiteness of the strain energy function that ν_1 and ν_2 must have real positive parts. ν_3, of course, must be real. We shall assume that ν_1 and ν_2 are not equal. Two more parameters k_1 and k_2 are defined as

$$k_j = (c_{11}\nu_j - c_{44})/(c_{13}+c_{44}), \qquad (j = 1, 2). \tag{55}$$

We now define

$$z_j = z/\sqrt{\nu_j}, \qquad (j = 1, 2, 3). \tag{56}$$

It can be shown (e.g. [3]) that the equilibrium equations are satisfied if the displacement components are represented by three potential functions $\phi_1(r, \theta, z)$, $\phi_2(r, \theta, z)$, and $\psi(r, \theta, z)$, which are harmonic in the (r, θ, z) space:

$$\begin{aligned} c_{44}u_r &= \frac{\partial\phi_1(r,\theta,z_1)}{\partial r} + \frac{\partial\phi_2(r,\theta,z_2)}{\partial r} + \frac{\partial\psi(r,\theta,z_3)}{r\partial\theta}, \\ c_{44}u_\theta &= \frac{\partial\phi_1(r,\theta,z_1)}{r\partial\theta} + \frac{\partial\phi_2(r,\theta,z_2)}{r\partial\theta} - \frac{\partial\psi(r,\theta,z_3)}{\partial r}, \\ c_{44}u_z &= \frac{k_1\partial\phi_1(r,\theta,z_1)}{\partial z} + \frac{k_2\partial\phi_2(r,\theta,z_2)}{\partial z}. \end{aligned} \tag{57}$$

From equations (52) and (57) and the relationships between strains and displacements, the stress components may be expressed in terms of the potential functions ϕ_1, ϕ_2, and ψ.

The physical problem of a rigid sphere indenting an elastic half-space is the characteristic of torsionless axisymmetry, i.e., ϕ_1 and ϕ_2 are independent of the coordinate θ, and we may set $\psi = 0$. The displacement components are now

$$\begin{aligned} c_{44}u_r &= \frac{\partial\phi_1(r,z_1)}{\partial r} + \frac{\partial\phi_2(r,z_2)}{\partial r}, \\ c_{44}u_z &= \frac{k_1\partial\phi_1(r,z_1)}{\partial z} + \frac{k_2\partial\phi_2(r,z_2)}{\partial z}. \end{aligned} \tag{58}$$

The stresses are

$$\begin{aligned} \sigma_{rr}+\sigma_{\theta\theta} &= 2\left(\frac{1}{\nu_3} - \frac{1+k_1}{\nu_1}\right)\frac{\partial^2\phi_1(r,z_1)}{\partial z_1^2} + 2\left(\frac{1}{\nu_3} - \frac{1+k_2}{\nu_2}\right)\frac{\partial^2\phi_2(r,z_2)}{\partial z_2^2} \\ \sigma_{rr}-\sigma_{\theta\theta} &= \frac{2}{\nu_3}\left\{\frac{\partial^2}{\partial r^2}[\phi_1(r,z_1)+\phi_2(r,z_2)] - \frac{\partial}{r\partial r}[\phi_1(r,z_1)+\phi_2(r,z_2)]\right\} \\ \sigma_{zz} &= (1+k_1)\frac{\partial^2\phi_1(r,z_1)}{\partial z_1^2} + (1+k_2)\frac{\partial^2\phi_2(r,z_2)}{\partial z_2^2} \\ \sigma_{rz} &= \frac{1+k_1}{\sqrt{\nu_1}}\frac{\partial^2\phi_1(r,z_1)}{\partial r\partial z_1} + \frac{1+k_2}{\sqrt{\nu_2}}\frac{\partial^2\phi_2(r,z_2)}{\partial r\partial z_2} \\ \sigma_{\theta z} &= \sigma_{\theta r} = 0 \end{aligned} \tag{59}$$

B. *Hertzian contact, normal load*

We shall be concerned with the stress field due to a circular contact region on an elastic half-space carrying a spherical rigid punch providing Hertzian normal pressure. The boundary conditions on the plane $z = 0$ of the half-space $z > 0$ are

$$\sigma_{rz} = 0, \tag{60}$$

$$\sigma_{zz} = \begin{cases} -P(a^2-r^2)^{1/2}, & r < a \\ 0, & r > a. \end{cases} \tag{61}$$

The stress components $\sigma_{\theta z}$, $\sigma_{r\theta}$ are zero everywhere due to symmetry. The quantity P/a is equal to the maximum pressure, and is related to the normal load by the equation

$$P = 3W/(2\pi a^3). \tag{62}$$

The stress field vanishes at large distances from the region of contact.

It can be shown that equation (60) is satisfied if we set

$$\begin{aligned} \phi_1(r, z_1) &= \frac{P\sqrt{v_1}}{(1+k_1)(\sqrt{v_1}-\sqrt{v_2})} G(r, z_1), \\ \phi_2(r, z_2) &= \frac{P\sqrt{v_2}}{(1+k_2)(\sqrt{v_2}-\sqrt{v_1})} G(r, z_2). \end{aligned} \tag{63}$$

where $G(r, z)$ is a harmonic function.

Equation (61) is then reduced to

$$\left.\frac{\partial^2 G(r, z)}{\partial z^2}\right|_{z=0} = \begin{cases} -(a^2-r^2)^{\frac{1}{2}}, & r < a, \\ 0, & r > a. \end{cases} \tag{64}$$

The form of the expression for $G(r, z)$ may be deduced from the analysis in [6]. We shall record the function $G(r, z)$ and the derivatives necessary for finding the stress and displacement components. Defining

$$t = z + ia, \qquad \rho^2 = t^2 + r^2, \tag{65}$$

then

$$\begin{aligned} G(r, z) &= \frac{1}{2}\operatorname{Im}\left\{\left(\frac{t^3}{3} - iat^2 - \frac{1}{2}r^2 z\right)\log(\rho+t) - \frac{11}{18}\rho^3 + \frac{5}{6}r^2\rho + \frac{3iat\rho}{2} + \frac{1}{2}r^2 z\right\}, \\ \frac{\partial G(r, z)}{\partial z} &= \tfrac{1}{2}\operatorname{Im}\{(t^2 - 2iat - \tfrac{1}{2}r^2)\log(\rho+t) - \tfrac{3}{2}\rho t + 2ia\rho + \tfrac{1}{2}r^2\}, \\ \frac{\partial G(r, z)}{\partial r} &= \frac{r}{2}\operatorname{Im}\left\{-z\log(\rho+t) - \frac{tz}{\rho+t} + \frac{2t^2}{3(\rho+t)} + \frac{1}{2}z + \frac{2}{3}\rho\right\}, \\ \frac{\partial^2 G(r, z)}{\partial r^2} - \frac{1}{r}\frac{\partial G(r, z)}{\partial r} &= \operatorname{Im}\left\{\frac{\rho^2+\rho t+t^2}{3(\rho+t)} - \frac{iat}{\rho+t} - \frac{1}{2}z\right\}, \\ \frac{\partial^2 G(r, z)}{\partial z^2} &= \operatorname{Im}\{z\log(t+\rho) - \rho\}, \\ \frac{\partial^2 G(r, z)}{\partial z \partial r} &= \frac{r}{2}\operatorname{Im}\left\{-\log(\rho+t) + \frac{1}{2}\;\frac{2ia-t}{\rho+t}\right\}. \end{aligned} \tag{66}$$

It can be readily verified that $G(r, z)$ is a harmonic function and that equation (64) is satisfied.

From equations (59) and (64), we find that the stress components are

$$\sigma_{rr}+\sigma_{\theta\theta} = \frac{2P}{(\sqrt{\nu_1}-\sqrt{\nu_2})}\left\{\sqrt{\nu_1}\left(\frac{1}{\nu_3(1+k_1)}-\frac{1}{\nu_1}\right)\frac{\partial^2 G(r, z_1)}{\partial z_1^2} -\sqrt{\nu_2}\left(\frac{1}{\nu_3(1+k_2)}-\frac{1}{\nu_2}\right)\frac{\partial^2 G(r, z_2)}{\partial z_2^2}\right\}, \tag{67}$$

$$\sigma_{rr}-\sigma_{\theta\theta} = \frac{2P}{\nu_3(\sqrt{\nu_1}-\sqrt{\nu_2})}\left\{\frac{\sqrt{\nu_1}}{1+k_1}\left[\frac{\partial^2 G(r, z_1)}{\partial r^2}-\frac{\partial G(r, z_1)}{r\partial r}\right] -\frac{\sqrt{\nu_2}}{1+k_2}\left[\frac{\partial^2 G(r, z_2)}{\partial r^2}-\frac{\partial G(r, z_2)}{r\partial r}\right]\right\}, \tag{68}$$

$$\sigma_{zz} = \frac{P}{(\sqrt{\nu_1}-\sqrt{\nu_2})}\left\{\sqrt{\nu_1}\frac{\partial^2 G(r, z_1)}{\partial z_1^2}-\sqrt{\nu_2}\frac{\partial^2 G(r, z_2)}{\partial z_2^2}\right\}, \tag{69}$$

$$\sigma_{rz} = \frac{P}{(\sqrt{\nu_1}-\sqrt{\nu_2})}\left\{\frac{\partial^2 G(r, z_1)}{\partial z_1\partial r}-\frac{\partial^2 G(r, z_2)}{\partial z_2\partial r}\right\}. \tag{70}$$

C. *Tangential load*

We assume that the rigid sphere in its vertical load has provided a contact pressure distribution of $\sigma_{zz} = -P(a^2-r^2)^{1/2}$ within the circle $r = a$. If the sphere is in horizontal sliding motion, the frictional effect between the surface of the half-space and the sphere at $z = 0$ can be described as

$$\sigma_{yz} = 0, \tag{71}$$

$$\sigma_{xz} = \begin{cases} Pf(a^2-r^2)^{1/2}, & r < a, \\ 0, & r > a. \end{cases} \tag{72}$$

where f is the coefficient of friction. Since we have already calculated the effect of the vertical load in the previous section, we shall set, at $z = 0$

$$\sigma_{zz} = 0. \tag{73}$$

In this problem it is more convenient to use the rectangular coordinate system (x, y, z). We now write out the stress components in terms of the three potential functions $\phi_1(x, y, z_1)$, $\phi_2(x, y, z_2)$ and $\psi(x, y, z_3)$.

$$\sigma_{xx}+\sigma_{yy} = 2\left(\frac{1}{\nu_3}-\frac{1+k_1}{\nu_1}\right)\frac{\partial^2\phi_1(x, y, z_1)}{\partial z_1^2}+2\left(\frac{1}{\nu_3}-\frac{1+k_2}{\nu_2}\right)\frac{\partial^2\phi_2(x, y, z_2)}{\partial z_2^2}$$

$$\sigma_{xx}-\sigma_{yy} = \frac{2}{\nu_3}\left(\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}\right)[\phi_1(x, y, z_1)+\phi_2(x, y, z_2)] +\frac{4}{\nu_3}\frac{\partial^2\psi(x, y, z_3)}{\partial x\partial y}$$

$$\sigma_{xy} = \frac{2}{v_3}\frac{\partial^2}{\partial x \partial y}[\phi_1(x, y, z_1) + \phi_2(x, y, z_2)]$$

$$-\frac{1}{v_3}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)\psi(x, y, z_3)$$

$$\sigma_{zz} = (1+k_1)\frac{\partial^2 \phi_1(x, y, z_1)}{\partial z_1^2} + (1+k_2)\frac{\partial^2 \phi_2(x, y, z_2)}{\partial z_2^2}$$

$$\sigma_{xz} = \frac{(1+k_1)}{\sqrt{v_1}}\frac{\partial^2 \phi_1(x, y, z_1)}{\partial x \partial z_1} + \frac{(1+k_2)}{\sqrt{v_2}}\frac{\partial^2 \phi_2(x, y, z_2)}{\partial x \partial z_2}$$

$$+\frac{1}{\sqrt{v_3}}\frac{\partial^2 \psi(x, y, z_3)}{\partial y \partial z_3}$$

$$\sigma_{yz} = \frac{(1+k_1)}{\sqrt{v_1}}\frac{\partial^2 \phi_1(x, y, z_1)}{\partial y \partial z_1} + \frac{(1+k_2)}{\sqrt{v_2}}\frac{\partial^2 \phi_2(x, y, z_2)}{\partial y \partial z_2}$$

$$-\frac{1}{\sqrt{v_3}}\frac{\partial^2 \psi(x, y, z_3)}{\partial x \partial z_3}. \tag{74}$$

We now set

$$\phi_1(x, y, z_1) = \frac{fP\sqrt{(v_1 v_2)}}{(1+k_1)(\sqrt{v_1}-\sqrt{v_2})}\frac{\partial H(x, y, z_1)}{\partial x} \tag{75}$$

$$\phi_2(x, y, z_2) = \frac{fP\sqrt{(v_1 v_2)}}{(1+k_2)(\sqrt{v_2}-\sqrt{v_1})}\frac{\partial H(x, y, z_2)}{\partial x} \tag{76}$$

$$\psi(x, y, z_3) = fP\sqrt{v_3}\frac{\partial H(x, y, z_3)}{\partial y} \tag{77}$$

where $H(x, y, z)$ is harmonic in (x, y, z). It is readily shown that all the boundary conditions on $z = 0$ are satisfied if

$$\left.\frac{\partial^3 H(x, y, z)}{\partial z^3}\right|_{z=0} = \begin{cases} -(a^2-r^2)^{1/2}, & r < a, \\ 0, & r > a. \end{cases} \tag{78}$$

Noting the similarity between equations (78) and (79), the required potential functions can be deduced from the function $G(r, z)$. We find that

$$\frac{\partial H(x, y, z_1)}{\partial x} = xA(r, z)$$
$$\frac{\partial H(x, y, z_2)}{\partial y} = yA(r, z) \tag{79}$$

where

$$A(r, z) = \frac{1}{2}\,\mathrm{Im}\left\{\left(-\frac{t^2}{2}+\frac{r^2}{8}+iat\right)\log(\rho+t)+\frac{5t\rho}{8}+\frac{(4ia-t)(\rho^3-t^3)}{12r^2}-ia\rho+\frac{1}{4}t^2-\frac{iat}{2}\right\}. \tag{80}$$

Note that the derivatives of $A(r, z)$ are

$$\frac{\partial A(r,z)}{\partial z}=\frac{\partial G(r,z)}{r\partial r},\tag{81}$$

$$\begin{aligned}\frac{\partial A(r,z)}{\partial r}=\frac{1}{2}\operatorname{Im}\left\{\frac{r}{4}\log(\rho+t)+\frac{t(\rho-t)}{2r}-\frac{t(\rho^3-t^3)}{6r^3}-\frac{t^2\rho}{4r^2}\right.\\ \left.+\frac{r}{8}+\frac{iat}{r}-\frac{2ia(\rho^3-t^3)}{3r^3}\right\},\end{aligned}\tag{82}$$

$$\begin{aligned}\frac{\partial^2 A(r,z)}{\partial r^2}=\frac{1}{2}\operatorname{Im}\left\{\frac{1}{4}\log(\rho+t)+\frac{3}{8}+\frac{t}{4\rho}+\frac{t^2(\rho-t)(\rho+2t)}{2r^4}\right.\\ \left.-\frac{t\rho}{2r^2}+\frac{t^2(\rho^2+t^2)}{4\rho r^2}-\frac{iat}{r^2}+\frac{2ia(\rho^3-t^3)}{r^4}-\frac{2ia\rho}{r^2}\right\}.\end{aligned}\tag{83}$$

The stress components and displacement components can all be written out in terms of the derivatives of $A(r, z)$. The three displacement components become

$$\begin{aligned}c_{44}u_x=&\frac{fP\sqrt{(v_1v_2)}}{(1+k_1)(\sqrt{v_2}-\sqrt{v_1})}\left[A(r,z_1)+\frac{x^2}{r}\frac{\partial A(r,z_1)}{\partial r}\right]\\ &+\frac{fP\sqrt{(v_2v_1)}}{(1+k_2)(\sqrt{v_1}-\sqrt{v_2})}\left[A(r,z_2)+\frac{x^2}{r}\frac{\partial A(r,z_2)}{\partial r}\right]\\ &+fP\sqrt{v_3}\left[A(r,z_3)+\frac{y^2}{r}\frac{\partial A(r,z_3)}{\partial r}\right],\end{aligned}\tag{84}$$

$$c_{44}u_y=\frac{fP\sqrt{(v_1v_2)}}{(1+k_1)(\sqrt{v_2}-\sqrt{v_1})}\frac{xy}{r}\frac{\partial A(r,z_1)}{\partial r}+\frac{fP\sqrt{(v_2v_1)}}{(1+k_2)(\sqrt{v_1}-\sqrt{v_2})}\frac{xy}{r}\frac{\partial A(r,z_2)}{\partial r},\tag{85}$$

$$c_{44}u_z=\frac{fPk_1\sqrt{(v_1v_2)}x}{(1+k_1)(\sqrt{v_2}-\sqrt{v_1})}\frac{\partial A(r,z_1)}{\partial z_1}+\frac{fPk_2\sqrt{(v_1v_2)}x}{(1+k_2)(\sqrt{v_1}-\sqrt{v_2})}\frac{\partial A(r,z_2)}{\partial z_2}.\tag{86}$$

The stress components are:

$$\begin{aligned}\sigma_{xz}=&\frac{fP\sqrt{v_2}}{(\sqrt{v_2}-\sqrt{v_1})}\left\{\frac{\partial A(r,z_1)}{\partial z_1}+\frac{x^2}{r}\frac{\partial^2 A(r,z_1)}{\partial r\partial z_1}\right\}\\ &+\frac{fP\sqrt{v_1}}{(\sqrt{v_1}-\sqrt{v_2})}\left\{\frac{\partial A(r,z_2)}{\partial z_2}+\frac{x^2}{r}\frac{\partial^2 A(r,z_2)}{\partial r\partial z_2}\right\}\\ &+fP\left\{\frac{\partial A(r,z_3)}{\partial z_3}+\frac{y^2}{r}\frac{\partial^2 A(r,z_3)}{\partial r\partial z_3}\right\}.\end{aligned}\tag{87}$$

$$\sigma_{yz}=\frac{fP}{(\sqrt{v_2}-\sqrt{v_1})}\frac{xy}{r}\left\{\sqrt{v_2}\frac{\partial^2 A(r,z_1)}{\partial r\partial z_1}-\sqrt{v_1}\frac{\partial^2 A(r,z_2)}{\partial r\partial z_2}\right\}-\frac{fPxy}{r}\frac{\partial^2 A(r,z_3)}{\partial r\partial z_3},\tag{88}$$

$$\sigma_{zz}=\frac{fPx\sqrt{(v_1v_2)}}{(\sqrt{v_1}-\sqrt{v_2})}\left\{\frac{\partial^2 A(r,z_1)}{\partial z_1^2}-\frac{\partial^2 A(r,z_2)}{\partial z_2^2}\right\},\tag{89}$$

$$\sigma_{xx}+\sigma_{yy} = \frac{2fP\sqrt{(\nu_1\nu_2)}}{\nu_3(\sqrt{\nu_2}-\sqrt{\nu_1})}\left\{\frac{\nu_1-(1+k_1)\nu_3}{\nu_1(1+k_1)}\frac{\partial^2 A(r,z_1)}{\partial z_1^2}-\frac{\nu_2-(1+k_2)\nu_3}{\nu_2}\frac{\partial^2 A(r,z_2)}{\partial z_2^2}\right\}, \quad (90)$$

$$\sigma_{xx}+\sigma_{yy} = \frac{2fP\sqrt{(\nu_1\nu_2)}}{(\sqrt{\nu_2}-\sqrt{\nu_1})\nu_3}\left\{\frac{1}{1+k_1}\left[\frac{3xr^2-2x^3}{r^2}\frac{\partial A(r,z_1)}{\partial r}+\frac{x(x^2-y^2)}{r^2}\frac{\partial^2 A(r,z_1)}{\partial r^2}\right]\right.$$
$$\left.-\frac{1}{1+k_2}\left[\frac{3xr^2-2x^3}{r^2}\frac{\partial A(r,z_2)}{\partial r}+\frac{x(x^2-y^2)}{r^2}\frac{\partial^2 A(r,z_2)}{\partial r^2}\right]\right\} \quad (91)$$
$$+\frac{4fP}{\sqrt{\nu_3}}\left[\frac{xy^2}{r}\frac{\partial^2 A(r,z_3)}{\partial r^2}+\frac{x^3}{r^3}\frac{\partial A(r,z_3)}{\partial r}\right],$$

$$\sigma_{xy} = \frac{2fP\sqrt{(\nu_1\nu_2)}}{(\sqrt{\nu_2}-\sqrt{\nu_1})\nu_3}\left\{\frac{1}{1+k_1}\left[\frac{x^2y}{r}\frac{\partial^2 A(r,z_1)}{\partial r^2}+\frac{y^3}{r^3}\frac{\partial A(r,z_1)}{\partial r}\right]\right.$$
$$\left.-\frac{1}{1+k_2}\left[\frac{x^2y}{r}\frac{\partial A(r,z_2)}{\partial r^2}+\frac{y^3}{r^3}\frac{\partial A(r,z_2)}{\partial r}\right]\right\} \quad (92)$$
$$-\frac{fP}{\sqrt{\nu_3}}\left[\frac{3yr^2-2y^3}{r^2}\frac{\partial A(r,z_3)}{\partial r}+\frac{y(y^2-x^2)}{r^2}\frac{\partial^2 A(r,z_3)}{\partial r^2}\right].$$

D. *Numerical results*

Using the elastic constants of a number of crystals, we have computed the stress fields in these materials under Hertzian contact with a rigid sphere. We have plotted the maximum shear stress distribution for three representative cases. For the purpose of comparison we have also plotted in the shear stress distribution of an isotropic solid with Poisson's ratio equal to 0.3. The dimensions have been normalized: the maximum surface pressure is equal to unit pressure, and the radius of the circle of contact is of unit length.

Figure 11 shows the maximum shear stress distribution in the isotropic material. As is well known, the greatest value occurs at approximately 0.48 below the surface along the axis of symmetry.

When the material is β-quartz, the situation is very similar to the isotropic case shown in Fig. 12. The greatest value of maximum shear stress is 0.324 and lies approximately 0.43 below the surface along the axis of symmetry.

In Fig. 13, the material is a zinc crystal. The highest value of maximum shear stress occurs on a circle of radius 0.8 and depth 0.3 below the surface of contact. The magnitude of the maximum shear stress is 0.296. However, in this case there is also a local maximum at the surface at the edge of the circle of contact. The magnitude of this local maximum is 0.285.

4. DISCUSSION

The analysis for plane elasticity presented here is contained in the general formal treatment given by Galin. Galin did not realize that the terms $(q_1\mu_2-q_1\mu_2)/(\mu_1-\mu_2)$ and $(p_2-p_1)/(\mu_1-\mu_2)$ were always imaginary; he apparently thought that these terms were generally complex. As a result, Galin's formal solutions are much more complicated, and do not exhibit the features that the pressure distribution at the bounding surface are of the same form for isotropic and anisotropic materials, and that the stress functions are of the same form. In this paper, we have attempted to show that the anisotropic elasticity

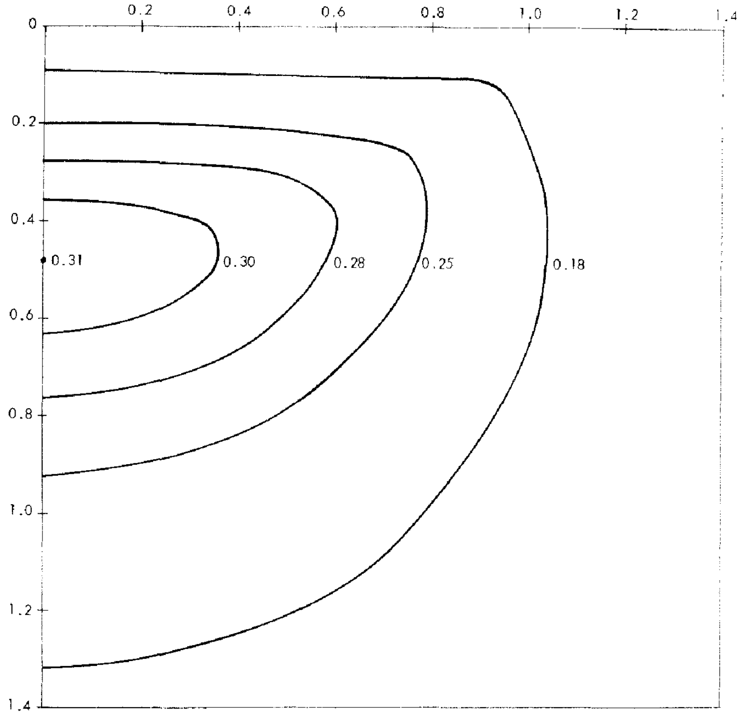

FIG. 11. Maximum shear stress distribution—isotropic solid.

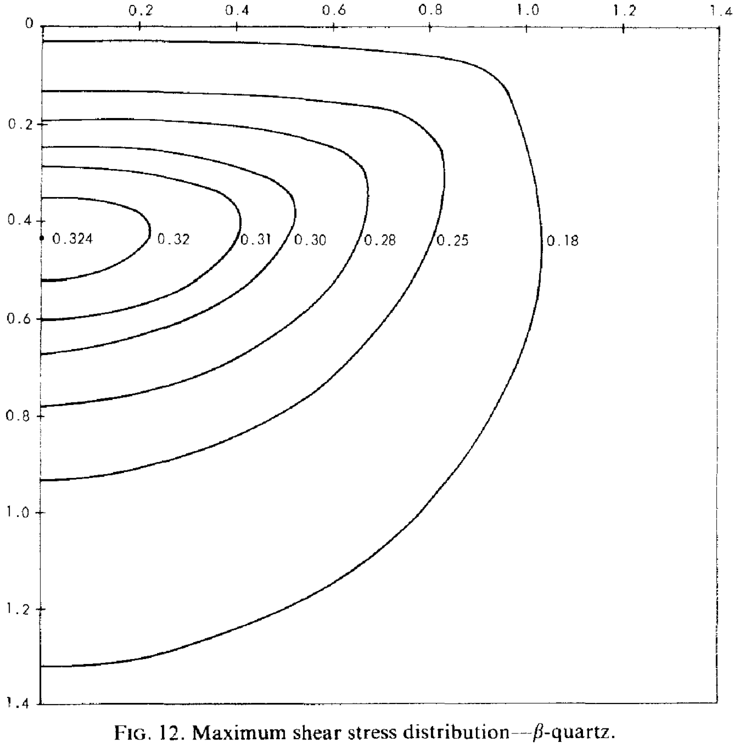

FIG. 12. Maximum shear stress distribution—β-quartz.

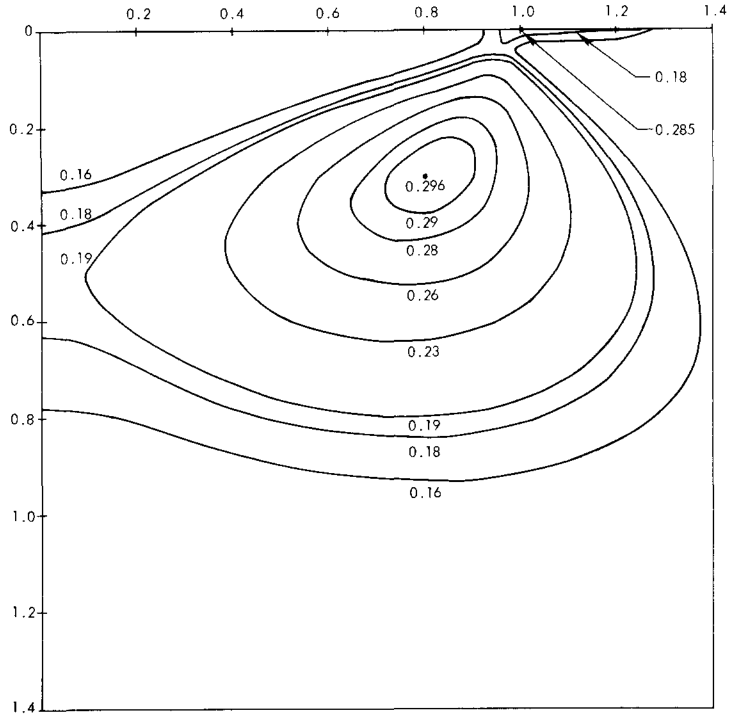

FIG. 13. Maximum shear stress distribution—zinc.

problem can be solved as easily as one would solve the corresponding isotropic problem. In fact, since the stress functions are the same, a knowledge of the solution to one leads immediately to the solution of the other. Green and Zerna considered the normal load case in a rather concise manner. They arrived at equation (20) and left further analysis and applications to physical problems to the readers. They did not consider the tangential load case.

Numerical results for the plane problem lead us to consider the three-dimensional problems. Recently, Conway, Ku, and Farnham [9], and Conway and Farnham [10] have presented numerical results of maximum shear stress distribution in a hexagonal crystal along the loading axis, under the implicit assumption that the maximum shear stress would occur along or near this axis.

The integral transform method employed in [9] and [10] is not suitable for finding stresses outside the loading zone. Following the work of Hamilton and Goodman [6], we have derived closed-form expressions applicable anywhere within the half-space. The numerical results indicate that the position of the maximum shear stress could be situated outside the vertical axis for some materials.

It is relevant to mention that Brilla [11] has considered some general punch problems for the elastic anisotropic half-plane, where the displacements u_x and u_y are both prescribed underneath the punch. Willis [12] has considered the more general situation of three-dimensional contact of general anisotropic bodies in which the contact area is an ellipse. Sveklo [13] has also analyzed the Boussinesq type problems for the anisotropic half-space.

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Абстракт—В работе рассматривается поле напряжений в контактной задаче Герца иараллельных цилиндров, изготовленных из некоторых общих анизотропных материалов и контактной задачи Герца поперечно изотропных сферических тел. Выводятся аналитические формулы, в явном виде, для компонентов напряжения для каждого случая. Численные результаты указывают, что распределение максимального напряжения сдвига может значительно отклоняться от подобного случая изотропии.